$h \in R$, $T: R[x]_{\leq 3} \to R_4$

$$T(x^2+1) = \begin{pmatrix} 1 \\ 0 \\ K \\ 1 \end{pmatrix}$$

$$T(x^3) = \begin{pmatrix} K \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$T(x^3+x+1) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T(x-1) = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

Scrivi $[M(T)]^{B}_{\mathcal{E}}$, $B = \{1, x, x^2, x^3\}$ base di $R[x]_{\leq 3}$

$\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ base standard di R^4

$$\left([T(1)]_{\mathcal{E}} , [T(x)]_{\mathcal{E}} , [T(x^2)]_{\mathcal{E}} , [T(x^3)] \right)$$

colonne della nostra matrice

Se $C = \{x^2+1, x^3, x^3+x+1, x-1\}$ è una base allora 1 può essere scritto come c.l. dei membri di C

$$1 = \lambda_1(x^2+1) + \lambda_2(x^3) + \lambda_3(x^3+x+1) + \lambda_4(x-1) = x^3(\lambda_2+\lambda_3) + x^2\lambda_1$$
$$+ x(\lambda_3+\lambda_4) + \lambda_1 + \lambda_3 + \lambda_4$$

$$\begin{cases} \lambda_2+\lambda_3 = 0 \\ \lambda_1 = 0 \\ \lambda_3+\lambda_4 = 0 \\ \lambda_1+\lambda_3-\lambda_4 = 1 \end{cases} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & -\frac{1}{2} \end{array}\right)$$

$$1 = -\frac{1}{2}(x^3) + \frac{1}{2}(x^3+x+1) - \frac{1}{2}(x-1)$$

$$t(1) = -\frac{1}{2}\,t(x^3) + \frac{1}{2}\,t(x^3+x+1) - \frac{1}{2}\,t(x-1) =$$

$$-\frac{1}{2}\begin{pmatrix}K\\0\\0\\1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\0\\1\\0\end{pmatrix} - \frac{1}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}-K-\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\end{pmatrix}$$

$$R[x]_{\leq 3} \xrightarrow{\sim} R^4$$

$$1 \longrightarrow e_1$$
$$x \longrightarrow e_2$$
$$x^2 \longrightarrow e_3$$
$$x^3 \longrightarrow e_4$$

$$t\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\\1\end{pmatrix} \qquad t\begin{pmatrix}0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}h\\0\\0\\1\end{pmatrix}$$

$$t\begin{pmatrix}1\\1\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \qquad t\begin{pmatrix}-1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

Quindi $C = \left\{ \begin{pmatrix}1\\1\\0\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\}$

$$\det\begin{pmatrix}1&1&0&1\\1&1&0&0\\1&0&0&0\\0&0&0&1\end{pmatrix} = 2 \neq 0$$ quindi C base

$$\left[r(t)\right]^{\mathcal{E}}_{\mathcal{E}} = \begin{pmatrix}0&1&1&K\\0&1&0&0\\1&0&K&0\\0&0&1&1\end{pmatrix}$$

$$C = \begin{pmatrix}1&-1&1&0\\1&1&0&0\\0&0&1&0\\1&0&0&1\end{pmatrix}$$

$$\begin{tikzcd}
(\mathbb{R}^4, \mathcal{E}) \arrow[r, "{[M(t)]^{\mathcal{E}}_{\mathcal{E}}}"] & (\mathbb{R}^4, \mathcal{E}) \\
(\mathbb{R}^4, C) \arrow[u, "C"] \arrow[r, "{[M(t)]^{C}_{C}}"'] \arrow[ur, bend left=60, "{[M(t)]^{C}_{\mathcal{E}}}"] & (\mathbb{R}^4, C) \arrow[u, "C"']
\end{tikzcd}$$

$$[M(t)]^{C}_{\mathcal{E}} = [M(t)]^{\mathcal{E}}_{\mathcal{E}} \cdot C \Rightarrow [M(t)]^{C}_{\mathcal{E}} \cdot C^{-1} = [M(t)]^{\mathcal{E}}_{\mathcal{E}}$$

$$C^{-1} = \left(\begin{array}{cccc|cccc} 1 & -1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow{G-J^{(\text{molto complicato})}}$$

$$\rightarrow \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 \end{array}\right)$$

$$\underbrace{\qquad\qquad}_{C^{-1}}$$

quindi $C^{-1} = \frac{1}{2} \begin{pmatrix} 1 & 1 & -1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & -1 & 1 & 2 \end{pmatrix}$

$$\frac{1}{2} \cdot \begin{pmatrix} 0 & 1 & 1 & K \\ 0 & 1 & 0 & 0 \\ 1 & 0 & K & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & -1 & 0 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & -1 & 1 & 2 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -1+K & 1-K & 3-K & 2K \\ -1 & 1 & 1 & 0 \\ 1 & 1 & -1+2K & 0 \\ 1 & -1 & 3 & 2 \end{pmatrix}$$

det di $\overbrace{\text{queste cose}}$ con laplace è $4K-4=0$

e quindi $K=1$

Per $K=1$ + non è invertibile

Per $K \neq 1$ + è invertibile

$$A = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 \end{pmatrix}$$

$$g(1) = \dim V = \dim \operatorname{Ker}(A - 1\,id_n) = \dim \operatorname{Ker} \begin{pmatrix} 1 & 1 & 1 & 1 \\ \vdots & & & \vdots \\ 1 & 1 & 1 & 1 \end{pmatrix} = 3$$

$$V_1 = < \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} >$$

$$V_5 = < \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} >$$

$A = \begin{pmatrix} 1 & t \\ 2 & 3 \end{pmatrix}$ $\quad v \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\quad$ quindi $A \cdot V = \lambda \cdot V = \begin{pmatrix} 2y \\ \lambda \end{pmatrix}$

$$\begin{pmatrix} 1 & t \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2+t \\ 7 \end{pmatrix}$$

$$\begin{cases} 2+t = 2\lambda \\ 7 = \lambda \end{cases} = \begin{cases} t = 12 \\ \lambda = 7 \end{cases}$$

Esercizio 18:

$$p(x) = (x^2+2x-3)(x^2-1) - (1+x)^2 - 2(x+3) + 4 + 2x(x+3)$$

$$= x^4 + 2x^3 - 3x^2 \rightarrow x^2(x^2+2x-3) \rightarrow x^2(x+3)(x-1)$$

$$x = 0, 0, -3, 1$$

$$\rightarrow \begin{pmatrix} 0 & & & 0 \\ & 0 & & \\ 0 & & -3 & \\ & & & 1 \end{pmatrix}$$